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Stability criteria for hierarchical triple systems

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Abstract In this paper, we give a summary of stability criteria that have been derived for hierarchical triple systems over the past few decades. We give a brief description and we discuss the criteria that are based on the generalisation of the concept of zero velocity surfaces of the restricted three body problem, to the general case. We also present criteria that have to do with escape of one of the bodies. Then, we talk about the criteria that have been derived using data from numerical integrations. Finally, we report on criteria that involve the concept of chaos. In all cases, wherever possible, we discuss advantages and disadvantages of the criteria and the methods their derivation was based on, and some comparison is made in several cases.

Keywords Celestial mechanics, Three body problem, Stability

1 INTRODUCTION

The three body problem is one of the most fascinating topics in mathematics and celestial mechanics. The basic definition of the problem is as follows: three point masses (or bodies of spherical symmetry) move in space, under their mutual gravitational attraction; given their initial conditions, we want to determine their subsequent motion.

Like many mathematical problems, it is not as simple as it sounds. Although the two body problem can be solved in closed form by means of elementary functions and hence we can predict the quantitative and qualitative behaviour of the system, the three body problem is a complicated nonlinear problem and no similar type of solution exists. More precisely, the

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former is integrable but the latter is not (if a system with n degrees of freedom has n independent first integrals in involution, then it is integrable; that is not the case for the three body problem).

One issue that is of great interest in the three body problem, is the stability (and instability) of triple systems. The stability (and instability) of triple systems is an intriguing problem which remains unsolved up to date. It has been a subject of study by many people, not only because of the intellectual challenge that poses, but also because of its importance in many areas of astronomy and astrophysics, e.g. planetary and star cluster dynamics.

In this work, we review the three body stability criteria that have been derived over the past few decades. We deal with the gravitational non-relativistic three-body problem and we concentrate on hierarchical triple systems. By hierarchical, we mean systems in which we can distinguish two different motions: two of the bodies form a binary and move around their centre of mass, while the third body is on a wider orbit with respect to the binary barycentre. This may not be the most strict definition of a hierarchical triple system (e.g. see Eggleton and Kiseleva 1995), but we use that one in order to cover as many triple system configurations as possible.

We would also like to point out that some of the criteria may apply to systems that are not hierarchical or they are marginally hierarchical (e.g. Wisdom's criterion for resonance overlap), according to the definition given in the previous paragraph. However, as they are related to other criteria that refer to hierarchical systems, we felt that we should mention them too.

2 STABILITY CRITERIA

There are two main types of stability criteria, depending on how they were derived: analytical and numerical. Following that classification, we are going to present the analytical criteria first and then we will discuss the criteria that have been derived from numerical integrations. Finally, we present criteria that are based on the concept of chaos.

Throughout the next paragraphs, we decided that it would be better if we kept the notation that each author used (with a few exceptions for the benefit of the reader).

2.1 Analytical Criteria

The derivation of analytical stability criteria in the three body problem has been dominated by the generalisation of the concept of surfaces of zero velocity of the restricted three-body problem, first introduced by Hill (1878a, 1878b, 1878c). It is known that in the circular restricted three body problem, there are regions in physical space where motion can and cannot occur. These regions are determined by means of the only known integral of the circular restricted problem, the so called Jacobi constant. This notion has been extended to the general three body problems by several authors: Golubev (1967, 1968a, 1968b), Saari (1974), who used an inequality similar to Sundman's, Marchal and Saari (1975), who used Sundman's inequality, Bozis (1976), who used

algebraic manipulations of the integrals of motion in the planar three body problem, Zare (1976, 1977), who made use of Hamiltonian dynamics; Saari (1984, 1987), who produced 'the best possible configurational velocity surfaces'. Also, Sergysels (1986), derived zero velocity surfaces for the general three dimensional three body problem, by using the method of Bozis (1976) and a rotating frame that does not take into account entirely the rotation of the three body system. Finally, Ge and Leng (1992) produced the same result as Saari (1987), using a modified version of the transformation given in Zare (1976). Easton (1971), Tung (1974) and Mialni and Nobili (1983) also discussed the topology of the restrictive surfaces.

The quantity c^2H , where c is the angular momentum and H is the energy of the three body system, controls the topology of the restrictive surfaces and it is the analog of the Jacobi constant of the circular restricted problem.

Szebehely (1977) and Szebehely and Zare (1977), using two body approximations, produced an expression for c^2H , which involved the masses, the semi-major axes and the eccentricities of the system. Then, that expression was compared with the critical value $(c^2H)_{crit}$ at the collinear Lagrangian points, which determine the openings and closings of the zero velocity surfaces. If the value of c^2H for a given triple configuration was smaller than the one at the inner Lagrangian point, then there could be no exchange of bodies, i.e the system was Hill stable. Although there was some discussion on the effect of the inclination, the derivation was for coplanar orbits.

Marchal and his collaborators (Marchal and Saari 1975, Marchal and Bozis 1982), produced a generalisation of the Hill curves to the general three dimensional three body problem by using the quantity ρ/ν as the controlling parameter of the restrictive surfaces, where ρ is the mean quadratic distance, ν is the mean harmonic distance and they are defined by the following equations:

$$M^* \rho^2 = m_1 m_2 r_{12}^2 + m_1 m_3 r_{13}^2 + m_2 m_3 r_{23}^2 \quad (1)$$

$$\frac{M^*}{\nu} = \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}, \quad (2)$$

where $M^* = m_1 m_2 + m_1 m_3 + m_2 m_3$ and r_{ij} is the distance between m_i and m_j .

Walker et al. (1980) derived the critical surfaces in terms of the parameters

$$\epsilon^{23} = \frac{m_1 m_2}{(m_1 + m_2)^2} \alpha_{23}^2 \quad \text{and} \quad \epsilon_{32} = \frac{m_3}{m_1 + m_2} \alpha_{23}^3$$

with $\alpha_{23} = \rho_2/\rho_3$ (ρ_2 is the distance between m_1 and m_2 , ρ_3 is the distance between the centre of mass of m_1 and m_2 , and m_3). ϵ^{23} measures the disturbance of m_3 by the binary, while ϵ_{32} is a measure of the disturbance of the binary by m_3 . Thus, for a given triple configuration, they evaluated the ϵ quantities and determined whether the system was Hill stable or not.

Walker and Roy (1981) investigated the effect that the eccentricities had on the stability limit, as the Walker et al. (1980) derivation applied only for coplanar, initially circular and corotational triple systems. They paid particular attention to the initial orbital phases of the system and they found that the critical value of $\alpha = \alpha_{cr}$ (α being the semi-major axis ratio of the two orbits) could be affected by up to 20%. Similar work was also done

in Valsecchi et al. (1984), but instead of using two body expressions for the angular momentum and energy of the system as Walker et al. (1980) did, they used the exact expressions; however the disagreement between the two methods was very small. This was also confirmed by Kiseleva et al. (1994b), who used the exact expressions for the angular momentum and energy to evaluate X_{SZ} (the critical initial semi-major axis for the Szebehely-Zare criterion). They found that their value was always larger by at most 5% compared to the one obtained by two body approximations.

Roy et al. (1984) computed the distance of the closest approach of m_2 to m_3 for a coplanar, corotational, hierarchical three body system (with $m_2 < m_1$ for the inner binary) and derived a condition for stability by manipulating the angular momentum and energy integrals. They ended up with the following inequality:

$$-2s \leq [\mu \frac{(1-k)^2}{1-\mu} + \frac{\mu_3}{1+\mu_3}] [\mu \frac{(1-\mu)^2}{1-k} + \frac{\mu\mu_3}{k} + \frac{\mu_3(1-\mu)^2}{1-k\mu}]^2, \quad (3)$$

where

$$s = \frac{c^2 H}{(m_1 + m_2)^5}, \quad \mu = \frac{m_2}{m_1 + m_2}, \quad \mu_3 = \frac{m_3}{m_1 + m_2},$$

k is defined by the relation $\rho_2(1-\mu) = \rho_3(1-k)$ (ρ_2 and ρ_3 are the magnitudes of the two Jacobian vectors of the hierarchical triple system) and it represents the distance of closest approach of m_2 to m_3 . If there exists a dynamical barrier between m_2 and m_3 , then, there will be values of k for which inequality (3) will not be satisfied. The largest of these values will give the measure of the closest approach of the two orbits. Their result was in agreement with the $c^2 H$ criterion.

The concept of Hill type surfaces that pose restrictions to the motion of three body systems, has also been used to study the motion in special cases.

Szebehely (1978), in the context of the circular restricted three body problem, derived a simple condition for a satellite to remain in orbit around the smaller primary in presence of the perturbations of the larger one. The condition is:

$$(\rho_2)_{max} \leq (\frac{\mu}{81})^{\frac{1}{3}}, \quad (4)$$

ρ_2 being the radius of the satellite circular motion around its primary m_2 and $\mu = m_2/(m_1 + m_2)$. The above condition is valid for both prograde and retrograde motion.

Markellos and Roy (1981) obtained a more accurate result for the same problem:

$$R_{max}^D = 1.4803(\frac{\mu}{81})^{\frac{1}{3}}[1 - 1.73(\frac{\mu}{81})^{\frac{1}{3}}] + O(\mu) \quad (5)$$

for prograde orbits and

$$R_{max}^D = 0.8428(\frac{\mu}{81})^{\frac{1}{3}}[1 - 0.55(\frac{\mu}{81})^{\frac{1}{3}}] + O(\mu) \quad (6)$$

for retrograde orbits, where R_{max}^D corresponds to $(\rho_2)_{max}$ of Szebehely (1978) and again, $\mu = m_2/(m_1 + m_2)$.

Walker (1983) investigated the Hill-type stability of a coplanar, with initially circular orbits, hierarchical three body system, where the total mass of the binary was small compared to the mass of the external body (e.g. satellite-planet-star). His results were in good agreement with Szebehely (1978) and Markellos and Roy (1981).

Donnison and Williams (1983, 1985) used the c^2H condition to determine the Hill stability of coplanar hierarchical three body systems with $m_1 \gg m_2, m_3$ (m_1 and m_2 form the inner binary). Using two body approximations for the angular momentum and the energy of the system and taking advantage of the fact that one of the masses was much greater than the other two, they concluded that their system was stable (in terms of exchange) when the following condition was satisfied:

$$e_{max}^2 \leq \frac{\lambda(\epsilon_1 - 3) + \lambda^2(\epsilon_2 - 3)}{1 + \lambda\epsilon_1 + \lambda^2\epsilon_2 + \lambda^3}, \quad (7)$$

where

$$\epsilon_1 = \left(\frac{a_1}{a_2}\right) \pm 2\left(\frac{a_2}{a_1}\right)^{\frac{1}{2}}, \epsilon_2 = \left(\frac{a_2}{a_1}\right) \pm 2\left(\frac{a_1}{a_2}\right)^{\frac{1}{2}}, \lambda = \frac{m_3}{m_2},$$

a_1 and a_2 are the semi-major axes of the inner and outer orbit respectively; the plus sign corresponds to prograde motion, while the minus sign to retrograde motion. Finally, e_{max} is the largest of either inner or outer eccentricity.

Donnison (1988), using the same approach mentioned above, investigated the stability of low mass binary systems moving on elliptical orbits in the presence of a large third mass, i.e. $m_3 \gg m_1 + m_2$.

Brasser (2002) dealt with systems where m_2 was smaller than the other two masses, which were of comparable size (m_1 and m_2 form the inner binary).

Gladman (1993), based on the work done by Marchal and Bozis (1982), produced analytical formulae for the critical separation Δ_c that two planets m_1 and m_2 , orbiting a star m_3 , should have in order to be Hill stable. He derived the following formulae (to lowest order):

(i) for initially circular orbits

$$\Delta_c \approx 2.40(\mu_1 + \mu_2)^{\frac{1}{3}} \quad (8)$$

(ii) equal mass planets, small eccentricities ($\mu_1 = \mu_2 = \mu$)

$$\Delta_c \approx \sqrt{\frac{8}{3}(e_1^2 + e_2^2) + 9\mu^{\frac{2}{3}}} \quad (9)$$

(iii) equal mass planets, equal but large eccentricities e

$$\Delta_c \approx 0.3e, \quad (10)$$

where $\mu_1 \equiv m_1/m_3$ and $\mu_2 \equiv m_2/m_3$ and e_1 and e_2 are the eccentricities of the inner and outer orbit respectively.

Veras and Armitage (2004), generalising Gladman's result, derived a criterion for two equal mass planets on initially circular inclined orbits to achieve Hill stability. They found that the planets were Hill stable if their initial separation was greater than

$$\Delta_{crit} = \epsilon + \eta \sqrt{\left(4 + \frac{\cos^2 I}{2}\right)(\epsilon + \chi \eta \mu^{\frac{2}{3}})} + [\chi \eta \mu^{\frac{2}{3}} - 3\eta^2 \mu \sqrt{\frac{4 + \frac{\cos^2 I}{2}}{\epsilon + \chi \eta \mu^{\frac{2}{3}}}}] + \dots, \quad (11)$$

where

$$\epsilon \equiv 2 + \cos^2 I - \cos I \sqrt{8 + \cos^2 I}, \quad \eta \equiv 1 - \frac{\cos I}{\sqrt{8 + \cos^2 I}},$$

$$\chi \equiv 3 \cdot 2^{\frac{2}{3}} \cdot 3^{\frac{1}{3}}, \quad \mu \equiv m/m_3,$$

m_3 is the mass of the star, m is the mass of the planets and I the inclination of the orbits.

Finally, in a series of papers, Donnison (1984a, 1984b, 2006) made use of the c^2H criterion to determine the stability of triple systems, where the outer body moved on a parabolic or hyperbolic orbit with respect to the centre of mass of the other two bodies. The first two papers dealt with coplanar systems, while the latest one examined systems with inclined orbits. In each paper, there was discussion about some special cases (equal masses and large m_1 in paper I, equal and unequal binary masses in paper II, equal masses, unequal binary masses, m_1 large in paper III; in all cases m_1 belonged to the inner binary).

The main disadvantage of the c^2H criterion is that it is a sufficient but not a necessary condition for stability. Exchange might not occur even when the condition is violated but it certainly cannot occur when the condition is satisfied. The lobes could also be open to infinity, but the bodies may or may not escape to infinity. Finally, things are not clear again when the third body is started outside (inside) the lobes, since the criterion cannot give any information whether the third body will escape or not from the system (will keep orbiting the binary or form a binary with one of the other masses).

The situation where one member of a triple system escapes to infinity was investigated by several authors. They derived sufficient conditions for the motion to be of hyperbolic-elliptic type, i.e. conditions for the distance between one body and the centre of mass of the two other bodies to increase indefinitely as time goes to infinity, while the distance between the other two bodies remains bounded. Such conditions can be found in Standish (1971), Yoshida (1972), Griffith and North (1973), Marchal (1974). Yoshida (1974) derived another criterion for hyperbolic-elliptic motion under the condition that the magnitude of the angular momentum of the three body system was above a certain level and Bozis (1981), in a paper closely related to the one of Yoshida (1974), he considered conditions for the smallest mass of a triple system to escape to infinity. Finally, a stronger escape criterion has been proposed by Marchal and his collaborators (Marchal et al. 1984a, 1984b). References to criteria before 1970, can be found in the above mentioned papers.

Usually, those criteria required that the distance ρ_0 and radial velocity $\dot{\rho}_0$ of the potential escaper (with respect to the barycentre of the binary formed by the other two bodies) were above certain values at some time t_0 . However, for large distances ρ , there is little difference between the criteria (Anosova 1986).

It should also be added here, that, in addition to the sufficient conditions for escape of one body, some of the above mentioned authors also gave sufficient conditions for ejection without escape; in such a situation, the ejected mass reaches a bounded distance and falls back toward the other two masses. Such conditions can be found in Standish (1972), Griffith and North (1973) and Marchal (1974).

2.2 Numerical integration criteria

The numerical work involves a wide range of simulations of triple systems. Several authors set up numerical experiments and investigated the orbital evolution of hierarchical triple systems.

Harrington (1972, 1975, 1977), in a series of papers, carried out numerical integrations of hierarchical triple systems with stellar and planetary mass ratios. In his first paper, he integrated equal mass systems with different initial conditions in order to determine their stability. He considered a system to be stable if there had been no change in the orbital elements during the period of integration, particularly in the semi-major axes or the eccentricities. The following situations were also defined as unstable: escape of one body, collision, i.e. two components got sufficiently close that it could be assumed that there were tidal or material interactions between the bodies involved, change to which bodies comprise the inner binary. A total of 420 orbits were integrated for 10 to 20 revolutions of the outer orbit. It was found that stability was insensitive to the eccentricity of the inner binary, for moderate eccentricity, to the argument of periastron of either orbit and to the mutual inclination of the two orbits (except when the inclination was within a few degrees of a perpendicular configuration). As a measure of stability, he used the quantity q_2/a_1 (q_2 was the outer periastron distance and a_1 the inner semi-major axis) and he found that stability existed above $q_2/a_1 = 3.5$ for prograde and $q_2/a_1 = 2.75$ for retrograde orbits. In his second paper, Harrington integrated coplanar systems with unequal masses (with the largest mass ratio never exceeding 100 : 1) and based on his numerical results, he derived the following limiting condition for stability:

$$q/a = [(q/a)_0 / \log 1.5] \log [1 + m_3/(m_1 + m_2)], \quad (12)$$

where q is the outer periastron distance, a is the inner semi major axis and $(q/a)_0$ is the parameter limit for equal masses. The above condition was improved in the last of the three papers, in which Harrington performed numerical simulations for systems which consisted of a stellar binary and a body of planetary mass (equation 12 does not apply in this case). The new empirical condition for stability was (regardless of which of the components

the planet was):

$$q_2/a_1 \geq A[1 + B \log \frac{1 + m_3/(m_1 + m_2)}{3/2}] + K. \quad (13)$$

A and B were determined empirically, with A being the limit on q_2/a_1 for the equal mass case and it was taken directly from the results of the first paper and B was then determined by a least square fit to the unequal mass cases; K is 0 if this is to be a mean fit and is approximately 2 if it is to be an upper limit. For coplanar prograde orbits, $A = 3.50$ and $B = 0.70$ and for retrograde, $A = 2.75$ and $B = 0.64$. Harrington also found that retrograde orbits were more stable than the prograde ones, a result which is in contrast with Szebehely's and Zare's predictions, as they found that prograde orbits were more stable than retrograde orbits. However, the results for equal masses and direct orbits were in good agreement, although Szebehely's results allow a slightly closer outer orbit. Of course, it should be borne in mind that the c^2H criterion is a sufficient stability condition, based on the possibility of exchange of bodies. It should also be pointed out that the definition of stability given by Harrington is a bit ambiguous. He classifies a triple system as stable if there is no "significant change" in the orbital elements during the period of integration, and particularly in the semi-major axes and eccentricities. Another point that raises some concern is that the integrations were performed for only 10 or 20 outer orbital periods. This could prove inadequate, although Harrington suggested that instabilities of this kind (exchange etc.) set in very quickly.

Graziani and Black (1981), in the context of planet formation and extra-solar planets, used numerical integrations to model planetary systems (star and two planets, which had the same mass in most of the numerical simulations) with prograde, coplanar and initially circular orbits. The systems were integrated for at least 100 revolutions of the longest period planet, or until instability was evident. The authors classified a system as unstable if there was clear evidence for secular changes in any of the orbits during the numerical integration. Based on their results, they obtained the following condition for stability:

$$\mu = 0.5 \frac{m_1 + m_2}{M_*} < \mu_{crit} = 0.175 \Delta^3 (2 - \Delta)^{-\frac{3}{2}}, \quad \mu \leq 1 \quad (14)$$

where the planets m_1 and m_2 orbit the star M_* . The parameter Δ gives the minimum initial separation between the companions in units of their mean distance from the central star, while μ is the mean mass of the two companions in units of the mass of the star. More specifically,

$$\Delta = 2 \frac{R - 1}{R + 1}, \quad R = \frac{R_2}{R_1}$$

with R_1 and R_2 being the semi-major axes of the inner and outer orbit respectively. Systems with $\mu \geq \mu_{crit}$ became unstable within a few tens of

planetary orbits. Black (1982) modified the above condition to apply for $\mu \geq 1$. The modified stability condition is:

$$\mu \leq \mu_{crit} = 0.083 \frac{\Delta^3}{(2 - \Delta)^3}. \quad (15)$$

Both the above stability conditions were confirmed by more integrations (Pendleton and Black 1983). However, equations (14) and (15) were in disagreement with equation (13), except a narrow range around $\mu = 1$.

Donnison and Mikulskis (1992) produced a modified version of equations (14) and (15), based on numerical integrations of circular, coplanar and prograde systems. A system was considered to be unstable when there was a change of more than 10% in either of the semi-major axes or/and either of the eccentricities altered by more than 0.1. Each numerical model was integrated for at least 1000 inner binary orbits or until the existence of instability was evident (which usually happened within the first 100 orbits). They derived the following values for μ_{crit} :

$$\mu_{crit} = 0.479 \frac{\Delta^3}{(2 - \Delta)^{\frac{3}{2}}}, \quad \mu \leq 1 \quad (16)$$

and

$$\mu_{crit} = 0.364 \frac{\Delta^3}{(2 - \Delta)^3}, \quad \mu \geq 1. \quad (17)$$

Donnison and Mikulskis (1994), following the same procedure as above, produced the following formulae for μ_{crit} in the case of retrograde orbits:

$$\mu_{crit} = 0.7692 \frac{\Delta^3}{(2 - \Delta)^{\frac{3}{2}}}, \quad \mu \leq 1 \quad (18)$$

and

$$\mu_{crit} = 0.6848 \frac{\Delta^3}{(2 - \Delta)^3}, \quad \mu \geq 1. \quad (19)$$

The results of Donnison and Mikulskis (1992, 1994) were in good agreement with the results of Black and his collaborators (for prograde orbits of course), but quite different from Harrington's results, except in the equal mass case. There was also agreement with the theory of Szebehely and Zare (1977), but only for prograde orbits.

Dvorak (1986) investigated the stability of P-type orbits in stellar binary systems, i.e. planet orbiting the binary system, in the context of the elliptic restricted three body problem. He performed numerical integrations of planets on initially circular orbits orbiting an equal mass binary system. The integration time span was 500 binary periods and a planetary orbit was classified as stable if its eccentricity remained smaller than 0.3 throughout the whole integration time. His results showed a region of stability far away from the primaries, a region of instability closer to the primaries and a chaotic (in the sense of unpredictability) zone between those two regions. This chaotic zone was limited by the lower critical orbit (LCO), defined as the largest unstable orbit for all starting positions of the planet, and the upper critical

orbit (UCO), defined as the orbit with the smallest semimajor axis for which the system was stable for all starting positions. A least squares parabolic fit to the numerical integration results yielded:

$$LCO = (2.09 \pm 0.30) + (2.79 \pm 0.53)e - (2.08 \pm 0.56)e^2 \quad (20)$$

$$UCO = (2.37 \pm 0.23) + (2.76 \pm 0.40)e - (1.04 \pm 0.43)e^2, \quad (21)$$

where e is the eccentricity of the primaries and the distance is given in AU. Each coefficient is listed along with its formal uncertainty. Although the above formulae were derived for systems where the primaries had equal masses, additional numerical integrations of P-type orbits in systems with unequal mass primaries (Dvorak et al. 1989) showed no dependence of the critical orbits on the mass ratio of the primaries. Finally, concerning P-type orbits, Pilat-Lohinger et al. (2003) investigated the stability of such orbits in three dimensional space. They integrated initially circular planetary orbits in equal mass binary systems, with a binary eccentricity varying from 0 to 0.5. The mutual inclination of the orbits was in the range $0^\circ - 50^\circ$. The orbits were classified as in Dvorak (1986), i.e. stable, chaotic and unstable, where stable meant that the planet did not suffer from a close encounter with one of the primaries for the whole integration time span (50000 periods of the primaries). It turned out that the inclination did not affect the stability limit significantly.

Rabl and Dvorak (1988), by using numerical integrations, established stability zones for S-type orbits in stellar binary systems (planet orbiting one of the stars of the binary system). The setup of their systems was similar to the one in Dvorak (1986), i.e. initially circular orbit for the massless particle and equal mass primaries. The maximum binary eccentricity considered was 0.6. An initially circular S-type orbit was classified as stable, if it remained elliptical with respect to its mother primary during the whole integration time of 300 periods of the primary bodies. Based on their results, they derived the following formulae:

$$LCO = (0.262 \pm 0.006) - (0.254 \pm 0.017)e - (0.060 \pm 0.027)e^2 \quad (22)$$

$$UCO = (0.336 \pm 0.020) - (0.332 \pm 0.051)e - (0.082 \pm 0.082)e^2, \quad (23)$$

where e is the eccentricity of the stellar binary. Note that the meaning of LCO and UCO is different compared to the P-type orbit case (the stable orbits lie inside LCO, while the unstable ones outside UCO). As in Dvorak (1986), the results showed the existence of a grey (chaotic) area between LCO and UCO. Pilat-Lohinger and Dvorak (2002) performed more numerical experiments on S-type orbits. Their models took into consideration varying binary mass ratios (0.1 – 0.9) and, besides a varying primary eccentricity, the planetary mass had an eccentricity from 0 to 0.5. The integration time was 1000 binary periods. They found that an increase in the eccentricities reduced the stability zone (the planetary eccentricity had less influence than the binary eccentricity, but it reduced the stability zone in a similar way). The results were also in agreement with the results of Rabl and Dvorak (1988). However, a quick inspection of the result tables in Pilat-Lohinger and Dvorak (2002), may suggest that the primary mass ratio has an effect on the stability zones, in contrast to what was mentioned above in the case of P-type orbits.

Holman and Wiegert (1999), also investigated the stability of P-type and S-type orbits in stellar binary systems. They performed numerical simulations of particles on initially circular and prograde orbits around the binary or around one of the stars, in the binary plane of motion and with different initial orbital longitudes. The binary mass ratio was taken in the range $0.1 \leq \mu \leq 0.9$ and the binary eccentricity in the range $0.0 \leq e \leq 0.7 - 0.8$. The integrations lasted for 10^4 binary periods. If a particle survived the whole integration time at all initial longitudes, then the system was classified as stable. Using a least squares fit to their data, they obtained: (i) for the inner region (S-type orbit):

$$a_c = [(0.464 \pm 0.006) + (-0.380 \pm 0.010)\mu + (-0.631 \pm 0.034)e + (0.586 \pm 0.061)\mu e + (0.150 \pm 0.041)e^2 + (-0.198 \pm 0.074)\mu e^2]a_b \quad (24)$$

(ii) for the outer region (P-type orbit):

$$a_c = [(1.60 \pm 0.04) + (5.10 \pm 0.05)e + (-2.22 \pm 0.11)e^2 + (4.12 \pm 0.09)\mu + (-4.27 \pm 0.17)e\mu + (-5.09 \pm 0.11)\mu^2 + (4.61 \pm 0.36)e^2\mu^2]a_b, \quad (25)$$

where a_c is the critical semi-major axis, a_b is the binary semi-major axis, e is the binary eccentricity and $\mu = m_2/(m_1 + m_2)$. Equation (24) is valid to 4% typically and to 11% in the worst case over the range of $0.1 \leq \mu \leq 0.9$ and $0.0 \leq e \leq 0.8$, while equation (25) is valid to 3% typically and to 6% in the worst case over the same ranges. An interesting finding was that, in the outer region, ‘islands’ of instability existed outside the inner stable region; this phenomenon was attributed to mean motion resonances and indicated that there was not a sharp boundary between stable and unstable regions. It should be mentioned here that equation (25), as presented in the paper of Holman and Wiegert, appears not to depend on a_b at all. However, this is probably a misprint, as equation (24) might suggest. The results of Holman and Wiegert are in good agreement with the results of Dvorak (1986) and Rabl and Dvorak (1988). Figures 1 demonstrate that agreement.

Kiseleva and her collaborators, performed numerical integrations of hierarchical triple systems with coplanar, prograde and initially circular orbits (Kiseleva et al. 1994a, 1994b). The mass ratios were within the range $1 : 1 - 100 : 1$. A system was classified as stable if it preserved its initial hierarchical configuration during the whole of the integration time span, which was normally 100 outer binary orbital periods, but certain cases were followed for 1000 or even for 10000 outer orbits (however, it appeared that the longer integration time had little effect on the stability boundary). These numerical calculations were later extended to eccentric binaries, inclined orbits (from 0° to 180°) and different initial phases, and an empirical condition for stability was derived (Eggleton and Kiseleva 1995):

$$Y_0^{min} \approx 1 + \frac{3.7}{q_{out}^{1/3}} - \frac{2.2}{1 + q_{out}^{1/3}} + \frac{1.4}{q_{in}^{1/3}} \frac{q_{out}^{1/3} - 1}{q_{out}^{1/3} + 1}, \quad (26)$$

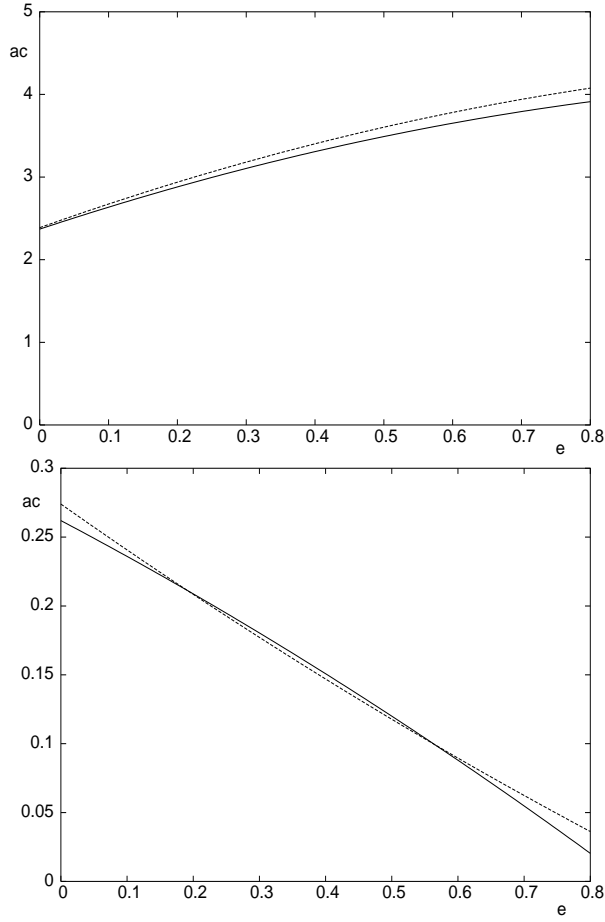


Fig. 1 Critical semi-major axis a_c against binary eccentricity e for a particle orbiting the binary. The top graph is for P-type orbits and the bottom one is for S-type orbits. The continuous lines comes from the results obtained from Dvorak (1986) and Rabl and Dvorak (1988), while the Holman-Wiebert results are shown with the dashed lines. For both graphs, $\mu = 0.5$ and the binary semi-major axis is 1 AU.

where Y_0^{min} is the critical initial ratio of the periastron distance of the outer orbit to the apastron distance of the inner orbit,

$$q_{in} = \frac{m_1}{m_2} \geq 1, \quad q_{out} = \frac{m_1 + m_2}{m_3}.$$

Y_0^{min} is related to the critical initial period ratio X_0^{min} by the following relation:

$$(X_0^{min})^{\frac{2}{3}} = \left(\frac{q_{out}}{1 + q_{out}}\right)^{\frac{1}{3}} \frac{1 + e_{in}}{1 - e_{out}} Y_0^{min}, \quad (27)$$

where e_{in} and e_{out} are the eccentricities of the inner and outer orbit respectively. The coefficients of equation (26) were obtained rather empirically ,

based on the numerical results that the authors had at their disposal. As for the effect of certain characteristics on the stability boundary, such as the orbital eccentricities, it was determined by the examination of a small number of mass ratios that the authors believed to be reasonably representative. The criterion appears to be reliable to about 20% for a wide range of circumstances, which is not very bad, considering the amount of parameters and the complex nature of the critical surface. It probably does not work very well in situations where there is a resonance or commensurability, but these are more common in systems with extreme mass ratios (e.g. star and planets), while the intention of the authors (as stated in their paper) was to investigate triple systems of comparable masses. It should be pointed out here that there is a misprint in formula (26) as given in Eggleton and Kiseleva (1995): the sign of the term $2.2/(1 + q_{out}^{1/3})$ is plus, while it should be minus (Aarseth 2003).

2.3 Chaotic Criteria

In the two previous sections, we presented stability criteria that were derived either analytically or based on results from numerical simulations. In this section, we discuss criteria that are based on the concept of chaos.

Wisdom (1980), applied the Chirikov resonance overlap criterion for the onset of stochastic behaviour (Chirikov 1979) to the planar circular restricted three body problem. He derived the following estimate of when resonances should start to overlap (the derivation holds for small eccentricities $e \leq 0.15$):

$$s_{overlap} \simeq 0.51\mu^{-2/7}, \quad (28)$$

where $\mu = m_2/(m_1 + m_2) \ll 1$. By using Kepler's third law, this can be expressed in terms of the semi-major axis separation as (Murray and Dermott 1999)

$$\Delta a_{overlap} \simeq 1.3\mu^{2/7}a_2, \quad (29)$$

where a_2 is the semi-major axis of the perturber. Hence, when the particle is in the region $a_2 \pm \Delta a_{overlap}$, the orbit is chaotic. A similar result to the one of Wisdom, was obtained through the use of a mapping, which was based on the approximation that perturbations to the massless body are localised near conjunction with the perturber (Duncan et al. 1989). It was found that

$$\Delta a_{overlap} \simeq 1.24\mu^{2/7}a_2, \quad (30)$$

which is in agreement with equation (29).

Mardling and Aarseth (1999) approached the stability problem in a different way, by noticing that stability against escape in the three body problem is analogous to stability against chaotic energy exchange in the binary-tides problem. The way energy and angular momentum are exchanged between the two orbits of a stable (unstable) hierarchical triple system is similar to the way they are exchanged in a binary undergoing normal (chaotic) tide-orbit

interaction. Having that in mind, they derived the following semi-analytical formula for the critical value of the outer pericentre distance R_p^{crit} :

$$R_p^{crit} = C \left[(1 + q_{out}) \frac{1 + e_{out}}{(1 - e_{out})^{\frac{1}{2}}} \right]^{\frac{2}{5}} \quad (31)$$

where $q_{out} = m_3/(m_1 + m_2)$ is the mass ratio of the outer binary and e_{out} is the outer binary eccentricity. If $R_p^{crit} \leq R_p^{out}$, then the system is considered to be stable. The above formula is valid for prograde and coplanar systems and it applies to escape of the outer body. C was determined empirically and it was found to be 2.8. A small heuristic correction of up to 30% was then applied for non-inclined orbits, to account for the increased stability (Aarseth and Mardling 2001, Aarseth 2004). Also, as stated in Aarseth and Mardling (2001), the criterion ignores a weak dependence on the inner eccentricity and inner mass ratio. Finally, we should mention here, that, numerical tests have showed that the criterion is working well for a wide range of parameters, but it has not been tested for systems with planetary masses so far (Aarseth 2004), probably because the authors were mainly interested in using the formula in star cluster simulations.

We would like to mention here that, Mardling (2007) has derived a resonance overlap criterion for the general three body problem.

We should point out, that the presence of chaos does not necessarily indicate instability, e.g. see Murray (1992), Gladman(1993). The reader should also recall the results of Dvorak (1986) and Rabl and Dvorak (1988), with the zones of unpredictability between the stable and unstable orbits. However, that kind of behaviour appears to depend on various parameters of the system, such as the mass ratios of the system. For example, Mudryk and Wu (2006), in their study of a planet orbiting one of the components of a stellar binary system, found little evidence of 'bound chaos' near the instability boundary (except in the case where the perturber is very small compared to the star, i.e. the case discussed by Gladman or covered by Wisdom's criterion) and as a result of that, they adopted the boundary of resonance overlap as the boundary of instability. That appears to be the case with Mardling and Aarseth too. A nice discussion in resonances and instability can be found in Mardling (2001).

3 Summary

We have attempted to collect and present the various criteria that have been derived for the stability of hierarchical triple systems over the past few decades.

Tables 1, 2 and 3 present the various criteria in a rather concise manner. Each Table consists of four columns, i.e. the 'Name' column, which gives the name of the relative paper(s), the 'Model/Restrictions' column, which gives a brief description of the systems for which the criterion is applicable (a blank line indicates that the criterion applies to the general case, without any restrictions), the 'Stability Type' column, which states what stability

means for a specific criterion and finally the 'Comments' column, where we give any extra information we consider important.

Table 1 lists the criteria that were derived analytically. Most of them were based on a generalisation of the concept of zero velocity surfaces of the circular restricted three body problem, with the quantity c^2H playing the role of the Jacobi constant. As stated in the corresponding section, the c^2H criterion is a sufficient condition and therefore, no conclusion can be drawn when it is violated. The Marchal and Bozis (1982) criterion is a good choice for one who intends to use a criterion from that specific category. However, depending on the system investigated, the other criteria could also be a useful alternative and even easier to apply. Table 1 also lists sufficient criteria for escape of one of the bodies. Although those criteria are not very useful on their own, because of their nature (they require some conditions to be satisfied at a moment t_0), they could be used as part of a computer code (e.g. for cluster simulations); however, their sufficient nature is a major disadvantage for that type of use.

Table 2 presents criteria that were based on results from numerical integrations. A task that is not particularly easy, as a triple system has many parameters to be taken into consideration (mass ratios and orbital parameters) and covering the whole of the parameter space at once is a rather difficult thing. Sometimes the various criteria were in agreement with each other, sometimes they were not. This can be attributed to many factors. The main one, in our opinion, is the different meaning that stability may have for different people. Szebehely (1984) gave 47 different definitions for stability in his 'Dictionary of Stability'. As the reader has probably noticed, almost each author mentioned in section (2.2), gave a different definition of what he considered as stable system. Another issue that raises concern is the integration time span. A system may appear to be stable for a certain time span, but becomes unstable when the integration is extended over longer timescales. Also, the choice of initial conditions may have an effect on the outcome. Finally, as stated in Kiseleva et al. (1994a), a matter of concern about those criteria is the fact that they involve instantaneous and not mean orbital parameters. The last two criteria of the table are probably the best from the numerical ones, the Eggleton-Kiseleva for stellar systems and the Holman-Wiegert for planets in binary systems (keep in mind that the planets are on initially circular orbits).

We would like to open a parenthesis here and mention that the stability of planets in binary systems is an area of research that is expected to become more and more important in the future, as there is an increasing number of exoplanets that are members of binary or multiple stellar systems (e.g. see Eggenberger et al. 2004). It appears that none of the above mentioned stability criteria, analytical or numerical, can cover the issue on its own. For instance, the planetary eccentricity is an important parameter not appearing in the criteria, although many exoplanets have eccentric orbits (of course most of the criteria were developed when none or very few exoplanets had been discovered by that time). Therefore, at the moment, one should choose the criterion (or a combination of different criteria) that fits the system he investigates better.

Table 1 Analytical criteria overview

Name	Model/Restrictions	Stability Type	Comments
Szebehely and Zare 1977	coplanar orbits	Hill	2b approx. for energy and ang. momentum
Marchal and Bozis 1982		Hill	limit cases discussed
Walker et al. 1980	coplanar, corotational init. circular orbits	Hill	series approx. for the controlling parameter
Roy et al. 1984	coplanar, corotational orbits	Hill	in agreement with the c^2H criterion
Szebehely 1978	circular restricted 3bp satellite-planet + star	Hill	
Markellos and Roy 1981	circular restricted 3bp satellite-planet + star	Hill	more accurate result than Szeb. 1978
Walker 1983	coplanar, init. circular orbits $m_3 \gg m_1 + m_2$	Hill	in agreement with Markellos and Roy
Donnison 1988	coplanar, init. circular orbits $m_3 \gg m_1 + m_2$	Hill	in agreement with the previous three papers
Donnison and Williams 1983, 1985	coplanar orbits $m_1 \gg m_2, m_3$	Hill	2b approx. for energy and ang. momentum
Gladman 1993	star + two planets init. circular orbits, equal planetary masses and small e , equal planetary masses and equal but large e	Hill	based on Marchal and Bozis 1982
Veras and Armitage 2004	star + two equal mass planets initially circular and inclined orbits	Hill	generalisation of Gladman's result
Donnison 1984a	coplanar, non-closed outer orbit equal masses large m_1	Hill	
Donnison 1984b	coplanar, non-closed outer orbit equal masses equal binary masses unequal binary masses	Hill	
Donnison 2006	non-coplanar parabolic outer orbit equal masses unequal binary masses large m_1	Hill	
Standish 1971		escape	
Yoshida 1972		escape	
Griffith and North 1973		escape	
Marchal 1974		escape	
Yoshida 1974		escape	
Bozis 1981	$m_1 \geq m_2 \geq m_3$	escape of m_3	
Marchal et al. 1984a, 1984b		escape	stronger than the previous relevant criteria

Table 2 Numerical integration criteria overview

Name	Model/Restrictions	Stability Type	Comments
Harrington 1977		no significant change in a, e , no escape no collision no change in hierarchy	inclination not important, $T_{int} = 10 - 20$ outer orbital periods
Graziani and Black 1981 (GB 1981)	Star + two planets, prograde, init. circular, coplanar orbits	Laplace	$T_{int} =$ at least 100 outer periods
Black 1982	$m_3 > m_1, m_2$, prograde, init. circular, coplanar orbits	Laplace	non-numerical, extends GB 1981
Donnison and Mikulskis 1992, 1994	same as GB 1981 plus retrograde orbits	change less than 10% in a and less than 0.1 in e	$T_{int} =$ at least 10^3 inner periods or until instability evident
Dvorak 1986 (DV 1986)	elliptic restricted P-type in equal mass stellar binaries, coplanar, init. circular plan. orbit	$e_{planet} < 0.3$	$T_{int} = 500$ bin. periods
Rabl and Dvorak 1988	same as DV 1986, but for S-type orbits	planet elliptic with respect to mother prim.	$T_{int} = 300$ bin. periods
Holman and Wiegert 1999	elliptic restricted P,S-type in stellar bin. coplanar, init. circular plan. orbit	planet survives at all init. longit. for T_{int}	$T_{int} = 10^4$ bin. periods
Eggleton and Kiseleva 1995	stellar mass ratios	change in hierarchy	mostly $T_{int} =$ 100 outer periods

Table 3 Chaotic criteria overview

Name	Model/Restrictions	Stability Type	Comments
Wisdom 1980	planar circular restricted, $e_{part.} \leq 0.15$	res. overlap	
Duncan et al. 1989	same as Wisdom 1980	res. overlap	agrees with Wisdom 1980
Mardling and Aarseth 1999	stellar systems, copl. orbits	res. overlap and escape	semi-analytical criterion

Finally, Table 3 lists criteria that involve the concept of chaos. In that context, instability in a three body system was thought to be the consequence of the overlap of sub-resonances within mean motion resonances. It was also mentioned that the presence of chaos in some cases, would not necessarily indicate instability.

We hope that this work can serve as a useful guide for anyone interested in the issue of the stability of hierarchical triple systems.

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References

- Aarseth, S.J.: Gravitational N-Body Simulations: Tools and Algorithms. Cambridge University Press, Cambridge pp. 295 (2003)
- Aarseth, S.J.: Formation and evolution of hierarchical systems. *RevMexAA* **21**, 156-162 (2004)
- Aarseth, S.J., Mardling, R.A.: The formation and evolution of multiple systems. In: Podsiadlowski P., Rappaport S., King A. R., D’Antona F., Burderi L. (eds) *Evolution of Binary and Multiple star systems*. ASP Conference Series, Vol. 229, pp. 77-88 (2001)
- Anosova, J.P.: Dynamical evolution of triple systems. *Ap & SS* **124**, 217-241 (1986)
- Black, D.C.: A simple criterion for determining the dynamical stability of three-body systems. *AJ* **87**, 1333-1337 (1982)
- Bozis, G.: Zero velocity surfaces for the general planar three-body problem. *Ap & SS* **43**, 355-368 (1976)
- Bozis, G.: Escape of the smallest mass of a triple system. *PASJ* **33**, 67-75 (1981)
- Brasser, R.: Hill stability of a triple system with an inner binary of large mass ratio. *MNRAS* **332**, 723-728 (2002)
- Chirikov, B.V.: Universal instability of many-dimensional oscillator systems. *Phys. Rep.* **52**, 263-379 (1979)
- Donnison, J.R.: The stability of masses during three-body encounters. *Cel. Mech.* **32**, 145-162 (1984a)
- Donnison, J.R.: The stability of binary star systems during encounters with a third star. *MNRAS* **210**, 915-927 (1984b)
- Donnison, J.R.: The effects of eccentricity on the hierarchical stability of low-mass binaries in the three-body systems. *MNRAS* **231**, 85-95 (1988)
- Donnison, J.R.: The Hill stability of a binary or planetary system during encounters with a third inclined body. *MNRAS* **369**, 1267-1280 (2006)
- Donnison, J.R., Mikulskis, D.F.: Three-body orbital stability criteria for circular orbits. *MNRAS* **254**, 21-26 (1992)
- Donnison, J.R., Mikulskis, D.F.: Three-body orbital stability criteria for circular retrograde orbits. *MNRAS* **266**, 25-30 (1994)
- Donnison, J.R., Williams, I.P.: The stability of coplanar three-body systems with applications to the solar system. *Cel. Mech.* **31**, 123-128 (1983)
- Donnison, J.R., Williams, I.P.: The hierarchical stability of satellite systems. *MNRAS* **215**, 567-573 (1985)
- Duncan, M., Quinn, T., Tremaine, S.: The long-term evolution of orbits in the solar system - A mapping approach. *Icarus* **82**, 402-418 (1989)
- Dvorak, R.: Critical orbits in the elliptic restricted three-body problem. *A & A* **167**, 379-386 (1986)
- Dvorak, R., Froeschlé, Ch., Froeschlé, Cl.: Stability of outer planetary orbits

- (P-types) in binaries. *A & A* **226**, 335-342 (1989)
- Easton, R.: Some topology of 3-body problems. *J. Differ. Equations* **10**, 371-377 (1971)
- Egenberger, A., Udry, S., Mayor, M.: Statistical properties of exoplanets III. Planet properties and stellar multiplicity. *A & A* **417**, 353-360 (2004)
- Eggleton, P., Kiseleva, L.: An empirical condition for stability of hierarchical triple systems. *ApJ* **455**, 640-645 (1995)
- Ge, Y., Leng, X.: An alternative deduction of the Hill-type surfaces of the spatial 3-body problem. *Cel. Mech. Dynam. Astron.* **53**, 233-254 (1992)
- Gladman, B.: Dynamics of systems of two close planets. *Icarus* **106**, 247-263 (1993)
- Golubev, V.G.: Regions where motion is impossible in the three body problem. *Doklady. Akad. Nauk. SSSR* **174**, 767-770 (1967)
- Golubev, V.G.: Hill stability in the unrestricted three-body problem. *Soviet Phys. Dokl.* **13**, 373-375 (1968a)
- Golubev, V.G.: Hill stability in the unbounded three-body problem. *Doklady. Akad. Nauk. SSSR*, **180**, 308-311 (1968b)
- Graziani, F., Black, D.C.: Orbital stability constraints on the nature of planetary systems. *ApJ* **251**, 337-341 (1981)
- Griffith, J.S., North, R.D.: Escape or retention in the three body problem. *Cel. Mech.* **8**, 473-479 (1974)
- Harrington, R.S.: Stability criteria for triple stars. *Cel. Mech.* **6** 322-327 (1972)
- Harrington, R.S.: Production of triple stars by the dynamical decay of small stellar systems. *AJ* **80**, no. 12, 1081-1086 (1975)
- Harrington, R.S.: Planetary orbits in binary stars. *AJ* **82**, no. 9, 753-756 (1977)
- Hill, G. W.: Researches in the lunar theory. *Am. J. Math.* **1**, 5-26 (1878a)
- Hill, G. W.: Researches in the lunar theory. *Am. J. Math.* **1**, 129-147 (1878b)
- Hill, G. W.: Researches in the lunar theory. *Am. J. Math.* **1**, 245-261 (1878c)
- Holman, M.J., Wiegert, P.A.: Long-term stability of planets in binary systems. *AJ* **117**, 621-628 (1999)
- Kiseleva, L.G., Eggleton, P.P., Anosova, J.P.: A note on the stability of hierarchical triple stars with initially circular orbits. *MNRAS* **267**, 161-166 (1994a)
- Kiseleva, L.G., Eggleton, P.P., Orlov, V.V.: Instability of close triple systems with coplanar initial doubly circular motion. *MNRAS* **270**, 936-946 (1994b)
- Marchal, C.: Sufficient conditions for hyperbolic-elliptic escape and for 'ejection without escape' in the three body problem. *Cel. Mech.* **9**, 381-393 (1974)
- Marchal, C., Bozis, G.: Hill stability and distance curves for the general three-body problem. *Cel. Mech.* **26**, 311-333 (1982)
- Marchal, C., Saari, D.G.: Hill regions for the general three-body problem. *Cel. Mech.* **12**, 115-129 (1975)
- Marchal, C., Yoshida, J., Yi-Sui, S.: A test of escape valid even for very small mutual distances I. The acceleration and the escape velocities of the third body. *Cel. Mech.* **33**, 193-207 (1984a)
- Marchal, C., Yoshida, J., Yi-Sui, S.: Three-body problem. *Cel. Mech.* **34**, 65-93 (1984b)
- Mardling, R.A.: Stability in the general three-body problem. In: Podsiadlowski P., Rappaport S., King A. R., D'Antona F., Burderi L. (eds) *Evolution of Binary and Multiple star systems*. ASP Conference Series, Vol. 229, pp. 101-116 (2001)
- Mardling, R.A.: Resonance, chaos and stability in the general three-body problem. In Vesperini E., Giersz M., Sills A. (eds) *Dynamical Evolution of Dense Stellar Systems*. IAU Symposium 246, 2007, preprint.
- Mardling, R.A., Aarseth, S.J.: Dynamics and stability of three-body systems. In: Steves B.A., Roy A.E. (eds) *The Dynamics of Small Bodies in the Solar System, A Major Key to Solar System Studies*, NATO ASI, vol. 90, pp. 385-392 Kluwer, Dordrecht (1999)
- Markellos, V.V., Roy, A.E.: Hill stability of satellite orbits. *Cel. Mech.* **23**, 269-275 (1981)
- Milani, A., Nobili, A.M.: On topological stability in the general three-body problem. *Cel. Mech.* **31**, 213-240 (1983)

-
- Mudryk, L.R., Wu, Y.: Resonance overlap is responsible for ejecting planets in binary systems. *ApJ* **639**, 423-431 (2006)
- Murray, C.D.: Solar system - wandering on a leash. *Nature* **357**, 542-543 (1992)
- Murray C. D., Dermott S. F.: *Solar System Dynamics*. Cambridge University Press, Cambridge (1999)
- Pendleton, Y.J., Black, D.C.: Further studies on criteria for the onset of dynamical instability in general three-body systems. *AJ* **88**, no. 9, 1415-1419 (1983)
- Pilat-Lohinger, E., Dvorak, R.: Stability of S-type orbits in binaries. *Cel. Mech. Dynam. Astron.* **82**, 143-153 (2002)
- Pilat-Lohinger, E., Funk, B., Dvorak, R.: Stability limits in double stars - A study of inclined planetary orbits. *A & A* **400**, 1085-1094 (2003)
- Rabl, G., Dvorak, R.: Satellite-type planetary orbits in double stars: a numerical approach. *A & A* **191**, 385-391 (1988)
- Roy, A.E., Carusi, A., Valsecchi, G.B., Walker, I.W.: The use of the energy and angular momentum integrals to obtain a stability criterion in the general hierarchical three-body problem. *A & A* **141**, 25-29 (1984)
- Saari, D.G.: Restrictions on the motion of the three-body problem. *SIAM J. APPL. MATH.* **26**, No. 4, 806-815 (1974)
- Saari, D.G.: From rotations and inclinations to zero configurational velocity surfaces. I - A natural rotating coordinate system. *Cel. Mech.* **33**, 299-318 (1984)
- Saari, D.G.: From rotations and inclinations to zero configurational velocity surfaces, II. The best possible configurational velocity surfaces. *Cel. Mech.* **40**, 197-223 (1987)
- Sergysels, R.: Zero velocity hypersurfaces for the general three-dimensional three-body problem. *Cel. Mech.* **38**, 207-214 (1986)
- Standish, E.M.Jr.: Sufficient conditions for escape in the three-body problem. *Cel. Mech.* **4**, 44-48 (1971)
- Standish, E.M.Jr.: Sufficient conditions for return in the three-body problem. *Cel. Mech.* **6**, 352-355 (1972)
- Szebehely, V.: Analytical determination of the measure of stability of triple stellar systems. *Cel. Mech.* **15**, 107-110 (1977)
- Szebehely, V.: Stability of artificial and natural satellites. *Cel. Mech.* **18**, 383-389 (1978)
- Szebehely, V.: Review of concepts of stability. *Cel. Mech.* **34**, 49-64 (1984)
- Szebehely, V., Zare, K.: Stability of classical triplets and of their hierarchy. *A&A* **58**, 145-152 (1977)
- Tung, C.C.: Some properties of classical integrals of general problem of three bodies. *Scientia Sinica* **17**, 306-330 (1974)
- Valsecchi, G.B., Carusi, A., Roy, A.E.: The effect of orbital eccentricities on the shape of the Hill-type analytical stability surfaces in the general three-body problem. *Cel. Mech.* **32**, 217-230 (1984)
- Veras, D., Armitage, P.J.: The dynamics of two massive planets on inclined orbits. *Icarus* **172**, 349-371 (2004)
- Walker, I.W.: On the stability of close binaries in hierarchical three-body systems. *Cel. Mech.* **29**, 215-228 (1983)
- Walker, I.W., Emslie, A.G., Roy, A.E.: Stability criteria in many-body systems I. An empirical stability criterion for co-rotational three body systems. *Cel. Mech.* **22**, 371-402 (1980)
- Walker, I.W., Roy, A.E.: Stability criteria in many-body systems II. On a sufficient condition for the stability of coplanar hierarchical three body systems. *Cel. Mech.* **24**, 195-225 (1981)
- Wisdom, J.: The resonance overlap criterion and the onset of stochastic behavior in the restricted three-body problem. *AJ* **85**, no. 8, 1122-1133 (1980)
- Yoshida, J.: Improved criteria for hyperbolic-elliptic motion in the general three-body problem. *PASJ* **24**, 391-408 (1972)
- Yoshida, J.: Improved criteria for hyperbolic-elliptic motion in the general three-body problem. II. *PASJ* **26**, 367-377 (1974)
- Zare, K.: The effects of integrals on the totality of solutions of dynamical

systems. Cel. Mech. **14**, 73-83 (1976)

Zare, K.: Bifurcation points in the planar problem of three bodies. Cel. Mech. **16**, 35-38 (1977)